

# On the Stability of Axisymmetric MHD Modes

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*Dedicated to Adrienne in memory of Pee-Pee-Islands*

The stability of axisymmetric ideal MHD equilibria which are symmetric with respect to the equatorial plane is considered. It is found that for external axisymmetric modes which are antisymmetric with respect to the equatorial plane and for profiles such that the current density vanishes at the free plasma boundary the stability problem reduces to a classical interior-exterior scalar eigenvalue problem. Because of the separation property the resulting stability condition is necessary and sufficient and is thus more stringent than criteria derived by choosing special test functions, e.g. the vertical shift condition.

## Introduction

An axisymmetric equilibrium is described by the Lüst-Schlüter equation [1–3]

$$\Delta_* \Psi + F = 0, \quad F(R, \Psi) = I' I + R^2 p',$$

$$\Delta_* = \frac{\partial^2}{\partial R^2} - \frac{1}{R} \frac{\partial}{\partial R} + \frac{\partial^2}{\partial z^2} = R^2 \nabla \cdot \frac{\nabla}{R^2}$$

for the flux function  $\Psi$ . Here,  $R, \phi, z$  are cylindrical coordinates,  $p(\Psi)$  is the pressure, and  $B_\phi = I(\Psi)/R$  is the toroidal field. The poloidal field  $B_p$  is related to  $\Psi$  by

$$B_p = \nabla \phi \times \nabla \Psi.$$

The prime indicates differentiation with respect to  $\Psi$ .

It has been shown in [4], [5] that, after minimization with respect to the  $\phi$ -component of  $\xi$ , the energy principle [6], [7] for axially symmetric displacements ( $\partial/\partial\phi = 0$ ) can be reduced to the form

$$\partial W_f \geq \frac{1}{2} \int_{V_f} \frac{d^3 \tau}{R^2} \left\{ |\nabla \xi|^2 + |f|^2 + \gamma p R^2 |\operatorname{div} \xi|^2 - \frac{\partial F}{\partial \Psi} |\xi|^2 \right\}, \quad (1)$$

where

$$\xi = \xi \cdot \nabla \Psi.$$

With the notation

$$\langle \dots \rangle = \int \dots \frac{d^2 S}{|\nabla \Psi|} \quad (2)$$

for integrals over the magnetic surface  $\Psi = \text{const}$  the quantity  $f$  is defined by

$$f = f(\Psi) = I \frac{\frac{d}{d\Psi} \left\langle \frac{\xi}{R^2} \right\rangle}{\left\langle \frac{1}{R^2} \right\rangle}. \quad (3)$$

Expression (1) was originally derived for the case of a rigid wall at the axisymmetric plasma boundary  $S_f$ . Let us take  $\Psi = 0$  on  $S_f$  and  $\Psi > 0$  in the plasma region  $V_f$ . Then, choosing the profile functions  $p(\Psi)$ ,  $I(\Psi)$  such that

$$I'(0) = p'(0) = 0 \quad (4)$$

makes (1) also valid for the case where the plasma is surrounded by a vacuum region  $V_v$  [7]. The choice (4) of the profile functions implies that the current density vanishes everywhere on  $S_f$ . If the current density were not zero there, the system could then be unstable with respect to peeling modes [8], [9]. The peeling mode is a non-axisymmetric mode localized near the free surface. Conditions (4) guarantee that the peeling criterion and, furthermore, even a local sufficient condition with respect to all modes [10] is satisfied near the free surface.

It is well-known (see, for instance, [8]) that for the case of non-vanishing shear the  $\gamma$  term in (1) can be minimized to give

$$\int_{V_f} p |\operatorname{div} \xi|^2 d^3 \tau \geq \int_{V_f} p |g|^2 d^3 \tau,$$

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where in the axisymmetric case  $g$  is the surface quantity:

$$g = g(\Psi) = \frac{\frac{d}{d\Psi} \langle \xi \rangle}{\langle 1 \rangle}. \quad (5)$$

So the axisymmetric stability problem is completely described by the scalar quantity  $\xi$ :

$$\delta W_f \geq \frac{1}{2} \int_{V_f} \frac{d^3\tau}{R^2} \left\{ |\nabla \xi|^2 + |f|^2 + \gamma p R^2 |g|^2 - \frac{\partial F}{\partial \Psi} |\xi|^2 \right\}. \quad (6)$$

If the pressure vanishes on  $S_f$ , the energy variation of the system can be written in the form [7]

$$\begin{aligned} \delta W &= \delta W_f + \delta W_v, \\ \delta W_v &= \frac{1}{2} \int_{V_v} d^3\tau \tilde{\mathbf{B}}^2, \quad \tilde{\mathbf{B}} = \nabla \times \mathbf{A}. \end{aligned} \quad (7)$$

The displacement vector  $\xi$  of the fluid disturbance is related to the single-valued vector potential  $\mathbf{A}$  of the magnetic field disturbance  $\tilde{\mathbf{B}}$  in the vacuum region by the condition [7]

$$\mathbf{n} \times \mathbf{A} = -(\mathbf{n} \cdot \xi) \mathbf{B} \quad \text{on } S_f, \quad (8)$$

while on the perfectly conducting outer wall  $S_w$  (which is assumed to be given with axial symmetry) one has

$$\mathbf{n} \times \mathbf{A} = 0 \quad \text{on } S_w. \quad (9)$$

For axisymmetric disturbances the vector fields can be divided into a toroidal and a poloidal part. The equilibrium field, for instance, can be written as

$$\mathbf{B} = \nabla \phi \times \nabla \Psi + I \nabla \phi.$$

The toroidal component of (8) yields

$$\mathbf{A} \cdot (\nabla \phi \times \nabla \Psi) = -\xi \frac{I}{R^2} \quad \text{on } S_f \quad (10)$$

and thus determines the poloidal part  $\mathbf{A}_p$  of  $\mathbf{A}$ .  $\delta W_v$  is minimized with respect to  $\mathbf{A}_p$  by the choice

$$\mathbf{A}_p = \nabla a,$$

which is compatible with (10) if  $a$  is permitted to be a multi-valued function increasing in the poloidal angle. Thus, only the toroidal part

$$\mathbf{A}_t = \xi \nabla \phi. \quad (11)$$

contributes to  $\delta W_v$ .

Equations (7), (9), and (11) together with the poloidal component of (8) yield

$$\xi = \xi \quad \text{on } S_f, \quad (12)$$

$$\delta W_v = \frac{1}{2} \int_{V_v} \frac{d^3\tau}{R^2} |\nabla \xi|^2, \quad (13)$$

and

$$\xi = 0 \quad \text{on } S_w. \quad (14)$$

### Up-down Symmetry

Suppose now that  $\Psi$  is symmetric with respect to the equatorial plane  $z = 0$ :

$$\Psi(R, z) = \Psi(R, -z). \quad (15)$$

Then, by putting

$$\xi = \eta + \zeta$$

with

$$\eta(R, z) = -\eta(R, -z), \quad \zeta(R, z) = \zeta(R, -z),$$

the energy variation  $\delta W_f$  (as well as the vacuum contribution  $\delta W_v$ ) is separated, yielding

$$\begin{aligned} \delta W_f &\geq \delta W_f^{(a)} + \delta W_f^{(s)}, \\ \delta W_f^{(a)} &= \frac{1}{2} \int_{V_f} \frac{d^3\tau}{R^2} \left\{ |\nabla \eta|^2 - \frac{\partial F}{\partial \Psi} |\eta|^2 \right\}, \end{aligned} \quad (16)$$

$$\delta W_f^{(s)} = \frac{1}{2} \int_{V_f} \frac{d^3\tau}{R^2} \left\{ |\nabla \zeta|^2 + |f|^2 + \gamma p R^2 |g|^2 - \frac{\partial F}{\partial \Psi} |\zeta|^2 \right\}, \quad (17)$$

$$f = I \frac{\frac{d}{d\Psi} \left\langle \frac{\zeta}{R^2} \right\rangle}{\left\langle \frac{1}{R^2} \right\rangle}, \quad g = \frac{\frac{d}{d\Psi} \langle \zeta \rangle}{\langle 1 \rangle},$$

because all coefficients in the functional (6) are symmetric in  $z$ . Note that the functional (16) contains neither  $f$  nor  $g$  because the antisymmetric part  $\eta$  cancels in (3) and (5). So, if the disturbance  $\eta$  leads to an instability, this instability cannot be stabilized either by increasing  $\gamma$  or by increasing the toroidal field.

If  $\eta$  is varied with the normalization

$$\int_{V_f} \frac{d^3\tau}{R^2} |\eta|^2 = 1 \quad (18)$$

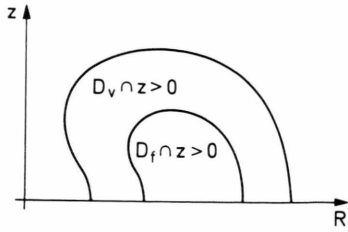


Fig. 1. Qualitative sketch of the fluid and vacuum regions.

and with the boundary conditions

$$\eta \text{ continuous across } S_f \cap z > 0, \quad (19)$$

$$\eta = 0 \text{ on } (S_w \cap z > 0) \cup [(V_f \cup V_v) \cap z = 0], \quad (20)$$

then the minimum of  $\delta W^{(a)}$  is attained for the lowest eigenvalue  $v_0$  of the interior-exterior eigenvalue problem

$$A_* \eta + \frac{\partial F}{\partial \Psi} \eta + v \eta = 0 \text{ in } D_f \cap z > 0, \quad (21)$$

$$A_* \eta = 0 \text{ in } D_v \cap z > 0, \quad (22)$$

$$\eta, \frac{\partial \eta}{\partial n} \text{ continuous across } \partial D_f \cap z > 0, \quad (23)$$

$$\eta = 0 \text{ on } [\partial D \cap z > 0] \cup [D \cap z = 0], \quad (24)$$

where  $D_f, D_v$  are the intersections of  $V_f, V_v$ , respectively, with the poloidal plane  $\phi = \text{const}$  (see Fig. 1) and  $D = D_f \cup D_v$ .

The condition

$$v_0 \geq 0 \quad (25)$$

is necessary and sufficient for the stability of axisymmetric disturbances which are antisymmetric with respect to the poloidal plane. The test function  $\eta = \partial \Psi / \partial z$  in  $D_f$  describes the vertical shift and is in general not minimizing.

The corresponding equations for  $\zeta$  are derived in Appendix A.

### The Free-Surface problem

Suppose that  $S_w$  is a magnetic surface, implying that  $\Psi$  is constant there:

$$\Psi = \alpha < 0, \text{ say, on } \partial D. \quad (26)$$

Suppose further that there are no singularities (wires) in  $V_v$ . Then, in addition to the equation

$$A_* \Psi + F = 0 \text{ in } D_f \quad (27)$$

we have

$$A_* \Psi = 0 \text{ in } D_v. \quad (28)$$

On  $\partial D_f$  it is then required that

$$\Psi, \frac{\partial \Psi}{\partial n} \text{ be continuous across } \partial D_f \quad (29)$$

in order that the magnetic field be continuous on  $S_f$ .

The total toroidal plasma current  $J$  is usually prescribed as a normalization:

$$\begin{aligned} J &= - \int_{D_f} \frac{1}{R} A_* \Psi \, dR \, dz = - \int_{\partial D_f} \mathbf{B} \cdot d\mathbf{l}, \\ &= - \int_{\partial D} \mathbf{B} \cdot d\mathbf{l} = - \int_{\partial D} \frac{\partial \Psi}{\partial n} \frac{dl}{R}. \end{aligned} \quad (30)$$

The mathematical problem described by (27)–(30) has been treated by several authors (see, for instance, [11], [12]). The solution of the non-linear boundary value problem determines the free boundary  $\partial D_f$ . Under relatively mild conditions on the profile functions  $I, p(\Psi)$  there exists a solution. The corresponding numerical problem is treated in [13].

Once  $\partial D_f$  has been determined, it can be considered as given for the stability problem.

### Large Aspect Ratio

Let  $R_{\min}$  and  $R_{\max}$  be the smallest and largest values of  $R$  in  $D_v$ , respectively. Then coordinates  $x, y$  are introduced by

$$\begin{aligned} R &= \bar{R} + \bar{r} x, \quad z = \bar{r} y, \\ \bar{r} &= \frac{1}{2}(R_{\max} - R_{\min}), \quad \bar{R} = \frac{1}{2}(R_{\max} + R_{\min}), \\ &\quad -1 \leq x \leq 1. \end{aligned}$$

Introducing scaled quantities by

$$\kappa = \frac{\Psi}{\bar{R} \bar{r}}, \quad h = \frac{I}{\bar{R}}, \quad G = \bar{R} \bar{r} g \quad (31)$$

yields

$$\left( \frac{\partial^2}{\partial x^2} - \frac{\varepsilon}{1 + \varepsilon x} \frac{\partial}{\partial x} + \frac{\partial^2}{\partial y^2} \right) \kappa + \frac{dh}{d\kappa} h + (1 + \varepsilon x)^2 \frac{dp}{d\kappa} = 0$$

for the equilibrium equation, where  $\varepsilon = \bar{r}/\bar{R}$  is the inverse aspect ratio. The equations relevant to stability

are

$$\langle \dots \rangle = 2\pi \bar{r} \oint \dots \frac{1 + \varepsilon x}{(\kappa_x^2 + \kappa_y^2)^{\frac{1}{2}}} dl, \quad (32)$$

where the (dimensionless) variable  $l$  is the arc length in the  $x, y$ -plane along the curve  $\kappa = \text{const}$ , and

$$\begin{aligned} \delta W_f = & \frac{\pi}{\bar{R}} \int \frac{dx dy}{1 + \varepsilon x} \left\{ |\xi_x|^2 + |\xi_y|^2 \right. \\ & + h^2 \left| \frac{\frac{d}{d\kappa} \left\langle \frac{\xi}{(1 + \varepsilon x)^2} \right\rangle}{\left\langle \frac{1}{(1 + \varepsilon x)^2} \right\rangle} \right|^2 + \gamma p (1 + \varepsilon x)^2 |G|^2 \\ & \left. - \left[ \frac{d}{d\kappa} \frac{dh}{d\kappa} h + (1 + \varepsilon x)^2 \frac{d^2 p}{d\kappa^2} \right] |\xi|^2 \right\}, \quad (33) \end{aligned}$$

where

$$G = \frac{\frac{d}{d\kappa} \langle \xi \rangle}{\langle 1 \rangle}. \quad (34)$$

After dropping all terms containing  $\varepsilon$  one is left with the equations

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \kappa + \frac{dh}{d\kappa} h + \frac{dp}{d\kappa} = 0 \quad \text{in } D_f, \quad (35)$$

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \kappa = 0 \quad \text{in } D_v, \quad (36)$$

$$\langle \dots \rangle = 2\pi \bar{r} \oint \dots \frac{1}{(\kappa_x^2 + \kappa_y^2)^{\frac{1}{2}}} dl, \quad (37)$$

$$\begin{aligned} \delta W_f = & \frac{\pi}{\bar{R}} \int dx dy \left\{ |\xi_x|^2 + |\xi_y|^2 \right. \\ & \left. + (h^2 + \gamma p) |G|^2 - \left( \frac{d}{d\kappa} \frac{dh}{d\kappa} h + \frac{d^2 p}{d\kappa^2} \right) |\xi|^2 \right\}, \quad (38) \end{aligned}$$

$$\delta W_v = \frac{\pi}{\bar{R}} \int dx dy \{ |\xi_x|^2 + |\xi_y|^2 \} \quad (39)$$

for the straight geometry.

According to Appendix A the stability eigenvalue problem reads

$$\begin{aligned} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \xi + \xi \frac{d}{d\kappa} \left( \frac{dh}{d\kappa} h + \frac{dp}{d\kappa} \right) \\ + \frac{d}{d\kappa} (h^2 + \gamma p) G + \sigma \xi = 0 \quad \text{in } D_f, \\ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \xi = 0 \quad \text{in } D_v, \end{aligned}$$

$$\xi - \bar{\xi} = \frac{\partial \xi}{\partial n} - \frac{\partial \bar{\xi}}{\partial n} + \frac{h^2 G}{|\nabla \kappa|} = 0 \quad \text{on } \partial D_f,$$

$$\bar{\xi} = 0 \quad \text{on } \partial D,$$

where  $\sigma = \bar{r}^2 \mu$  is the eigenvalue and  $\partial/\partial n$  is the dimensionless normal derivative on  $\partial D_f$ .

### Circular Cylinder

If the lines  $\kappa = \text{const}$  are concentric circles, it is useful to introduce polar coordinates  $r, \theta$  by

$$x = \frac{r}{\bar{r}} \cos \theta, \quad y = \frac{r}{\bar{r}} \sin \theta, \quad dx dy = \frac{1}{\bar{r}^2} r dr d\theta.$$

One then has

$$\bar{r} \kappa' = -B_\theta, \quad \kappa_x^2 + \kappa_y^2 = \bar{r}^2 \kappa'^2 = B_\theta^2, \quad h = -B_z,$$

where the prime denotes the derivative with respect to  $r$  and

$$\xi = -\bar{R} \bar{\xi}_r B_\theta, \quad \langle \dots \rangle = \frac{2\pi r}{B_\theta} \int \dots d\theta. \quad (40)$$

Separating the  $\theta$ -dependence with  $\exp im\theta$  leads to

$$\begin{aligned} \delta W_f = 2\pi^2 \bar{R} \int_0^{r_f} r dr \left[ |(B_\theta \bar{\xi}_r)'|^2 + \frac{m^2}{r^2} B_\theta^2 |\bar{\xi}_r|^2 \right. \\ \left. + (\gamma p + B_z^2) \frac{|G|^2}{\bar{r}^2 \bar{R}^2} - B_\theta \left( \frac{p'}{B_\theta} + \frac{B_z' B_z}{B_\theta} \right)' |\bar{\xi}_r|^2 \right], \quad (41) \end{aligned}$$

$$\delta W_v = 2\pi^2 \bar{R} \int_{r_f}^\infty r dr \left[ |(B_\theta \bar{\xi}_r)'|^2 + \frac{m^2}{r^2} B_\theta^2 |\bar{\xi}_r|^2 \right]. \quad (42)$$

Here,  $r_f$  is the radius of the fluid and it is assumed for simplicity that the position of the wall is infinitely far away ( $\bar{r} \rightarrow \infty$ ). Because of conditions (4) and the equilibrium equation

$$p' + B_z' B_z + \frac{B_\theta}{r} (r B_\theta)' = 0 \quad (43)$$

the last term in the integrand of (41) can be transformed by partial integration to give together with the first term

$$\begin{aligned} \int_0^{r_f} r dr \left[ |(B_\theta \bar{\xi}_r)'|^2 - B_\theta \left( \frac{p'}{B_\theta} + \frac{B_z' B_z}{B_\theta} \right)' |\bar{\xi}_r|^2 \right] \\ = \int_0^{r_f} r dr \left[ B_\theta^2 |\bar{\xi}_r'|^2 - B_\theta^2 \frac{|\bar{\xi}_r|^2}{r^2} - (|\bar{\xi}_r|^2 B_\theta^2)' \frac{1}{r} \right] \\ = \int_0^{r_f} r dr \left[ B_\theta^2 \left| \bar{\xi}_r' - \frac{\bar{\xi}_r}{r} \right|^2 + \frac{1}{r} (2p + B_z^2)' |\bar{\xi}_r|^2 \right]. \quad (44) \end{aligned}$$

The two cases 1)  $m \neq 0$  and 2)  $m = 0$  now have to be distinguished.

1) Here, we have  $G \equiv 0$  and the minimum of the integral (42) is attained for

$$B_\theta \xi_r = C r^{-|m|}, \quad C = B_\theta(r_f) \xi_r(r_f) r_f^{|m|},$$

yielding

$$\delta W_v \geq 2\pi^2 \bar{R} |m| |\xi_r|^2 B_\theta^2(r_f).$$

Finally, using the second form of (44), one gets

$$\frac{1}{2\pi^2 \bar{R}} (\delta W_f + \delta W_v) \geq (|m| - 1) B_\theta^2 |\xi_r|^2 (r_f) + \int_0^{r_f} r dr B_\theta^2 \left[ |\xi_r'|^2 + (m^2 - 1) \frac{|\xi_r|^2}{r^2} \right]. \quad (45)$$

At the origin the test function  $\xi_r$  satisfies [14]:

$$\left. \begin{aligned} \xi_r'(0) &= 0 & \text{for } |m| = 1, \\ \xi_r(0) &= 0 & \text{for } |m| \neq 1. \end{aligned} \right\} \quad (46)$$

So, (45) shows the well-known property that it is positive for  $|m| > 1$  and zero for  $m = \pm 1$ ,  $\xi_r' = 0$ . The latter describes the marginal shift perturbation.

2) In the case  $m = 0$  it is found that

$$G = -\bar{r} \bar{R} \frac{(r \xi_r)'}{r}, \quad (47)$$

and it is useful to use third form of (44) by making another partial integration in the last term of the integrand and using the boundary conditions (46) and  $p(r_f) = 0$ . The result is

$$\begin{aligned} \frac{\delta W_f}{2\pi^2 \bar{R}} &= B_z^2 |\xi_r|^2 (r_f) + \int_0^{r_f} r dr \left[ (\gamma p + B_z^2) \left| \xi_r' + \frac{\xi_r}{r} \right|^2 \right. \\ &\quad \left. + B_\theta^2 \left| \xi_r' - \frac{\xi_r}{r} \right|^2 - \frac{1}{r} (2p + B_z^2) (\xi_r' \xi_r^* + \xi_r \xi_r'^*) \right] \\ &= B_z^2 |\xi_r|^2 (r_f) + \int_0^{r_f} r dr \left[ (\gamma - 1) p \left| \xi_r' + \frac{\xi_r}{r} \right|^2 \right. \\ &\quad \left. + B_z^2 \left( |\xi_r'|^2 + \frac{|\xi_r|^2}{r^2} \right) + (B_\theta^2 + p) \left| \xi_r' - \frac{\xi_r}{r} \right|^2 \right]. \quad (48) \end{aligned}$$

where the star denotes the complex conjugate. The last form of (48) exhibits stability for  $\gamma \geq 1$ ,  $p \geq 0$ .

For situations where the wall is infinitely far, any deviation from the circular cylinder is probably unstable. This has been shown for special cases [15]–[18] and will be treated elsewhere.

## Wall Stabilization for Linear Profiles

If the profiles  $I' I$  and  $p'$  are linear, conditions (4) imply that

$$I' I = R_{\min}^2 (1 - \beta_p) \lambda \Psi, \quad p' = \beta_p \lambda \Psi \quad (49)$$

with constants  $\lambda > 0$ ,  $\beta_p \geq 0$ . The functions (49) can be integrated to give

$$I^2 = R_{\min}^2 (1 - \beta_p) \lambda \Psi^2 + I_0^2, \quad p = \frac{1}{2} \beta_p \lambda \Psi^2. \quad (50)$$

For the case where the wall  $S_w$  coincides with the plasma surface  $S_f$ , stability can be demonstrated by means of a trick [5]. The ansatz

$$\xi = \alpha \Psi \quad (51)$$

with a finite function  $\alpha$  satisfies the rigid wall boundary condition and yields

$$\delta W_f \geq \frac{1}{2} \int_{V_f} \frac{d^3 \tau}{R^2} \{ \Psi^2 |\nabla \alpha|^2 + |f|^2 + \gamma p R^2 |g|^2 \} \quad (52)$$

after making appropriate use of the equilibrium equation. However, if the system is to remain stable when the wall is a finite distance away from the plasma surface, it has to be shown that functional (52) is positive definite:

$$\int_{V_f} \frac{d^3 \tau}{R^2} \{ \Psi^2 |\nabla \alpha|^2 + |f|^2 + \gamma p R^2 |g|^2 \} \geq C \int_{V_f} d^3 \tau \Psi^2 |\alpha|^2. \quad (53)$$

The validity of (53) with a positive constant  $C$  is proved in Appendix B. This means that the system remains stable if the wall is removed from the plasma by a small but finite amount. However, if the wall is removed too far away, there may be instabilities for non-convex curves  $\partial D_f$  [19]. The question whether there are toroidal configurations which are stable without any wall [20] is still open and will be treated in a forthcoming paper.

## Acknowledgement

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### Appendix A

Here, necessary conditions for the minimum of the functional

$$W = \frac{1}{2} \int_{V_f} \frac{d^3\tau}{R^2} (|\nabla\zeta|^2 + |f|^2 + \gamma p R^2 |g|^2 - \frac{\partial F}{\partial \Psi} |\zeta|^2) + \frac{1}{2} \int_{V_v} \frac{d^3\tau}{R^2} |\nabla\tilde{\zeta}|^2 \quad (A1)$$

with

$$f(\Psi) = I \frac{\frac{d}{d\Psi} \left\langle \frac{\zeta}{R^2} \right\rangle}{\left\langle \frac{1}{R^2} \right\rangle}, \quad g(\Psi) = \frac{\frac{d}{d\Psi} \langle \zeta \rangle}{\langle 1 \rangle},$$

$$\langle \dots \rangle = \int \dots \frac{d^2S}{|\nabla\Psi|}$$

are derived for test functions  $\zeta, \tilde{\zeta}$  subject to the boundary conditions

$$\zeta = 0 \quad \text{on the magnetic axis,} \quad (A2)$$

$$\tilde{\zeta} = \zeta \quad \text{on } S_f, \quad (A3)$$

$$\tilde{\zeta} = 0 \quad \text{on } S_w \quad (A4)$$

and the normalization

$$N = \frac{1}{2} \int_{V_f} \frac{d^3\tau}{R^2} |\zeta|^2 = 1. \quad (A5)$$

The variation of (A1) is of the form

$$\delta(W - \mu N) = \int_{V_f} \frac{d^3\tau}{R^2} \nabla\zeta \cdot \nabla\delta\zeta - \int_{V_f} \frac{d^3\tau}{R^2} \frac{\partial F}{\partial \Psi} \zeta \delta\zeta + \int_{V_f} \frac{d^3\tau}{R^2} (f\delta f + \gamma p R^2 g\delta g) + \int_{V_v} \frac{d^3\tau}{R^2} \nabla\tilde{\zeta} \cdot \nabla\delta\tilde{\zeta} - \mu \int_{V_f} \frac{d^3\tau}{R^2} \zeta \delta\zeta, \quad (A6)$$

where  $\mu$  is a Lagrangian multiplier and

$$\delta f = I \frac{\frac{d}{d\Psi} \left\langle \frac{\delta\zeta}{R^2} \right\rangle}{\left\langle \frac{1}{R^2} \right\rangle}, \quad \delta g = \frac{\frac{d}{d\Psi} \langle \delta\zeta \rangle}{\langle 1 \rangle}.$$

With the boundary conditions (A2)–(A4) taken into account, the first and fourth integrals in (A6) can be transformed by partial integration:

$$\int_{V_f} \frac{d^3\tau}{R^2} \nabla\zeta \cdot \nabla\delta\zeta + \int_{V_v} \frac{d^3\tau}{R^2} \nabla\tilde{\zeta} \cdot \nabla\delta\tilde{\zeta} = - \int_{V_f} \frac{d^3\tau}{R^2} \delta\zeta \Delta_* \zeta - \int_{V_v} \frac{d^3\tau}{R^2} \delta\tilde{\zeta} \Delta_* \tilde{\zeta} + \int_{S_f} d^2S \delta\zeta \frac{1}{R^2} \mathbf{n} \cdot \nabla(\zeta - \tilde{\zeta}), \quad (A7)$$

where  $\mathbf{n}$  is the outer normal on  $S_f$ . In order to rewrite the third integral of (A6), functions  $\Psi, \chi$  as defined in Appendix B serve as coordinates:

$$\int_{V_f} \dots d^3\tau = \int_{V_f} \dots 2\pi R \frac{d\Psi}{|\nabla\Psi|} \frac{d\chi}{|\nabla\chi|} = \int d\Psi \langle \dots \rangle. \quad (A8)$$

One then has

$$\begin{aligned} & \int_{V_f} \frac{d^3\tau}{R^2} (f\delta f + \gamma p R^2 g\delta g) \\ &= \int d\Psi \left( f I \frac{d}{d\Psi} \left\langle \frac{\delta\zeta}{R^2} \right\rangle + \gamma p g \frac{d}{d\Psi} \langle \delta\zeta \rangle \right) \\ &= - \int d\Psi \left[ (fI)_\Psi \left\langle \frac{\delta\zeta}{R^2} \right\rangle + \gamma (pg)_\Psi \langle \delta\zeta \rangle \right] + f I \left\langle \frac{\delta\zeta}{R^2} \right\rangle \Big|_{\Psi=0} \\ &= - \int_{V_f} \frac{d^3\tau}{R^2} [(fI)_\Psi + \gamma R^2 (pg)_\Psi] \delta\zeta \\ & \quad + \int_{S_f} d^2S \frac{fI}{R^2 |\nabla\Psi|} \delta\zeta, \end{aligned} \quad (A9)$$

where  $\Psi = p = 0$  on  $S_f$  and (A2) have been used. So, the variation (A6) contains volume as well as surface contributions:

$$\begin{aligned} & \delta(W - \mu N) \\ &= - \int_{V_f} \frac{d^3\tau}{R^2} \left[ \Delta_* \zeta + (fI + \gamma p R^2 g)_\Psi + \mu \zeta + \frac{\partial F}{\partial \Psi} \zeta \right] \delta\zeta \\ & \quad - \int_{V_v} \frac{d^3\tau}{R^2} \Delta_* \tilde{\zeta} \delta\tilde{\zeta} + \int_{S_f} \frac{d^2S}{R^2} \left( \frac{\partial \zeta}{\partial n} - \frac{\partial \tilde{\zeta}}{\partial n} + \frac{fI}{|\nabla\Psi|} \right) \delta\zeta. \end{aligned} \quad (A10)$$

For arbitrary  $\delta\zeta, \delta\tilde{\zeta}$  the variation (A10) does not vanish unless the Euler-Lagrange equations

$$\Delta_* \zeta + \frac{\partial F}{\partial \Psi} \zeta + (fI + \gamma p R^2 g)_\Psi + \mu \zeta = 0 \quad \text{in } V_f, \quad (A11)$$

$$\Delta_* \tilde{\zeta} = 0 \quad \text{in } V_v \quad (A12)$$

and, in addition to (A2)–(A4), the natural boundary condition

$$\frac{\partial \zeta}{\partial n} - \frac{\partial \tilde{\zeta}}{\partial n} + \frac{fI}{|\nabla\Psi|} = 0 \quad \text{on } S_f \quad (A13)$$

are satisfied. Note that in (A11) both partials with respect to  $\Psi$  are at constant  $R$ . Equations (A2)–(A4), (A11)–(A13) constitute an eigenvalue problem with  $\mu$  as the eigenvalue. The condition that the lowest eigenvalue  $\mu_0$  be non-negative,

$$\mu_0 \geq 0, \quad (A14)$$

is necessary and sufficient for stability against axisymmetric disturbances. For disturbances which are anti-symmetric with respect to the equatorial plane, the eigenvalue problem reduces to (21)–(24) of the main text.

## Appendix B

In order to prove that a finite constant  $C$  exists for relation (53), it is useful to distinguish the diamagnetic case 1)  $\beta_p \geq 1$  from the paramagnetic case 2)  $0 \leq \beta_p < 1$ .

1) Here, it is shown that

$$\int_{V_f} \Psi^2 \frac{d^3\tau}{R^2} \{ |\nabla\alpha|^2 + \frac{1}{2} \gamma \beta_p \lambda R^2 |g|^2 \} \geq C \int_{V_f} d^3\tau \Psi^2 |\alpha|^2 \quad (\text{B1})$$

with

$$g = \frac{1}{\langle 1 \rangle} \frac{d}{d\Psi} \langle \alpha \rangle \Psi. \quad (\text{B2})$$

The quantity  $\lambda$  can be considered as an eigenvalue of the linear problem

$$\begin{cases} \Delta_* \Psi + [R_{\min}^2 (1 - \beta_p) + R^2 \beta_p] \lambda \Psi = 0 & \text{in } D_f, \\ \Psi = 0 & \text{on } \partial D_f. \end{cases} \quad (\text{B3})$$

Standard variational methods lead to the estimate

$$\lambda \geq \frac{\pi^2}{4 R_{\max} (R_{\min} + R_{\max} \beta_p)} \left( \frac{1}{\bar{r}^2} + \frac{1}{\bar{z}^2} \right), \quad (\text{B4})$$

where  $2\bar{z}$  is the extension of  $D_f$  in the  $z$ -direction.

For the proof of (B1) the test function  $\alpha$  is divided in the following way:

$$\alpha = \bar{\alpha} + \tilde{\alpha}, \quad (\text{B5})$$

$$\bar{\alpha}(\Psi), \quad \langle \tilde{\alpha} \rangle = 0. \quad (\text{B6})$$

Introducing  $\Psi, \chi(R, z)$  as orthogonal coordinates with

$$0 \leq \chi \leq 2\pi, \quad \nabla\Psi \cdot \nabla\chi = 0, \quad d^3\tau = 2\pi R \frac{d\Psi}{|\nabla\Psi|} \frac{d\chi}{|\nabla\chi|},$$

one obtains

$$|\nabla\alpha|^2 = |\alpha_\Psi|^2 |\nabla\Psi|^2 + |\alpha_\chi|^2 |\nabla\chi|^2 \geq |\tilde{\alpha}_\chi|^2 |\nabla\chi|^2, \quad (\text{B7})$$

$$\langle \dots \rangle = 2\pi \int_0^{2\pi} \dots R \frac{d\chi}{|\nabla\Psi| |\nabla\chi|}, \quad \int_{V_f} \dots d^3\tau = \int d\Psi \langle \dots \rangle.$$

On a specified surface  $\Psi = \text{const}$  one has

$$\left\langle \frac{|\nabla\alpha|^2}{R^2} \right\rangle \geq C_1(\Psi) \left\langle \frac{|\tilde{\alpha}|^2}{R^2} \right\rangle. \quad (\text{B8})$$

The constant  $C_1$  in (B8) is determined by introducing a new variable  $\theta$  by

$$\frac{d\theta}{d\chi} = (2\pi)^2 R [\langle 1 \rangle |\nabla\Psi| |\nabla\chi|]^{-1}, \quad 0 \leq \theta \leq 2\pi. \quad (\text{B9})$$

One then obtains

$$\langle \dots \rangle = \frac{\langle 1 \rangle}{2\pi} \int_0^{2\pi} \dots d\theta, \quad \int_0^{2\pi} \tilde{\alpha} d\theta = 0, \quad (\text{B10})$$

$$\begin{aligned} \left\langle \frac{|\tilde{\alpha}_\chi|^2}{R^2} |\nabla\chi|^2 \right\rangle &= 2\pi \int_0^{2\pi} \frac{|\tilde{\alpha}_\chi|^2}{R} |\nabla\chi| \frac{d\chi}{|\nabla\Psi|} \\ &= (2\pi)^3 \langle 1 \rangle^{-1} \int_0^{2\pi} \frac{|\tilde{\alpha}_\theta|^2 d\theta}{|\nabla\Psi|^2} \\ &\geq (2\pi)^3 \langle 1 \rangle^{-1} (\max_\chi |\nabla\Psi|^2)^{-1} \int_0^{2\pi} |\tilde{\alpha}_\theta|^2 d\theta \\ &\geq (2\pi)^3 \langle 1 \rangle^{-1} (\max_\chi |\nabla\Psi|^2)^{-1} \int_0^{2\pi} |\tilde{\alpha}|^2 d\theta \\ &= C_1 \langle |\tilde{\alpha}|^2 \rangle, \end{aligned}$$

$$C_1 = (2\pi)^4 \langle 1 \rangle^{-2} (\max_\chi |\nabla\Psi|^2)^{-1}. \quad (\text{B11})$$

Now, by multiplying (B11) by  $\Psi^2$  and integrating over  $\Psi$  it is found that

$$(\text{B12})$$

$$\int_{V_f} \frac{d^3\tau}{R^2} \Psi^2 |\nabla\alpha|^2 \geq C_2 \int_{V_f} d^3\tau \Psi^2 |\tilde{\alpha}|^2, \quad C_2 = \min_\Psi C_1(\Psi).$$

For estimating the non-oscillating part, the  $|g|^2$  term is needed:

$$g = \frac{\Psi}{\langle 1 \rangle} \frac{d}{d\Psi} \eta, \quad \eta = \bar{\alpha} \langle 1 \rangle \Psi, \quad (\text{B13})$$

$$\begin{aligned} \int_{V_f} d^3\tau |g|^2 &= \int d\Psi \langle 1 \rangle^{-1} \Psi^2 \left| \frac{d\eta}{d\Psi} \right|^2 \\ &\geq (\max \langle 1 \rangle)^{-1} \int d\Psi \Psi^2 \left| \frac{d\eta}{d\Psi} \right|^2 \\ &\geq \frac{1}{4} (\max \langle 1 \rangle)^{-1} \int d\Psi |\eta|^2 \\ &= \frac{1}{4} (\max \langle 1 \rangle)^{-1} \int d\Psi |\bar{\alpha}|^2 \langle 1 \rangle^2 \Psi^2 \\ &\geq \frac{1}{4} \frac{\min \langle 1 \rangle}{\max \langle 1 \rangle} \int d\Psi |\bar{\alpha}|^2 \Psi^2 \langle 1 \rangle \quad (\text{B14}) \\ &= C_3 \int_{V_f} d^3\tau \Psi^2 |\bar{\alpha}|^2, \quad C_3 = \frac{1}{4} \frac{\min \langle 1 \rangle}{\max \langle 1 \rangle}. \end{aligned}$$

Relations (B12) and (B14) yield (B1) with

$$C = \min(C_2, \frac{1}{2} \gamma \beta_p \lambda C_3). \quad (\text{B15})$$

2) The proof for the paramagnetic case  $0 \leq \beta_p < 1$  is more complicated and will not be given here.

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